ON THE DIFFICULTY OF COMPUTING HIGHER-TWIST CORRECTIONS

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Abstract

We discuss the evaluation of power corrections to hard scattering and decay processes for which an operator product expansion is applicable. The Wilson coefficient of the leading-twist operator is the difference of two perturbative series, each of which has a renormalon ambiguity of the same order as the power corrections themselves, but which cancel in the difference. We stress the necessity of calculating this coefficient function to sufficiently high orders in perturbation theory so as to make the uncertainty of the same order or smaller than the relevant power corrections. We investigate in some simple examples whether this can be achieved. Our conclusion is that in most of the theoretical calculations which include power corrections, the uncertainties are at least comparable to the power corrections themselves, and that it will be a very difficult task to improve the situation.

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1 Introduction

In this paper we address the problem of controlling power corrections in effective theories. As an example consider e^+e^- annihilation into hadrons, for which the cross section is described by a perturbation series, computed at the parton level, plus power corrections which are proportional to the condensates of higher dimensional operators. In practice the value of the gluon condensate is obtained by comparing the experimental value of some quantity derived from $R_{e^+e^-}(Q^2)$ to its theoretical expression. This parameter is then used to predict many other physical quantities, such as form factors and decay constants. Given that only a few (typically one or two) terms of the perturbative series are known, and that the series are plagued by renormalon ambiguities, which are of the same order as the contribution from the condensate, one may wonder whether the value of the condensate is really known to sufficient accuracy to be used in other processes where power corrections are important for the theoretical predictions. This problem is not limited to the gluon condensate, but is also present for other important parameters of effective theories, such as the binding energy $(\overline{\Lambda})$ and kinetic energy (λ_1) of the Heavy Quark Effective Theory (HQET), and the matrix elements of higher-twist operators in deep inelastic scattering (DIS), such as those for the Gross-Llewellyn Smith and Bjorken sum rules. We argue that the problem is not solved at present, and that the uncertainties in the determination of these non-perturbative parameters are seriously underestimated.

The computation of power corrections requires the evaluation of the matrix elements of higher-twist or higher-dimensional operators. In addition, however, it also requires the calculation of the Wilson coefficient functions to sufficiently high order of perturbation theory for the cancellation of "renormalon ambiguities" to be under control [1]. The reason for this requirement is that these ambiguities are of the same order as the power corrections. Since, in the calculations performed up to now, only the first few terms of the perturbation series are known, it is not possible to check that the remaining terms are indeed negligible. By studying some simple examples, we will show that the knowledge of only a few terms is, in general, insufficient to control the power corrections. Although the results rely on some approximations which we are forced to adopt in these examples, it is likely that our conclusions will remain valid in general, and that in most cases a further theoretical effort is needed.

Our assumption throughout this paper is that one is attempting to evaluate the first power corrections (i.e. the next-to-leading twist contributions [2]) with an uncertainty that is smaller

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[1]We will show in section 2 that, in predictions for physical quantities, the Wilson coefficient functions can be written as linear combinations of two (or more) perturbation series, each of which has a high order behaviour such that its Borel transform has a renormalon singularity. The residue of the singularity cancels in the combination, however.

[2]We will frequently misuse the expression "higher-twist" to mean generic power corrections, and not just those to light-cone dominated quantities, such as deep inelastic structure functions.

than these corrections. The discussion can be readily generalized to include higher-order power corrections, with an increase in technical complexity, but following the same conceptual principles.

Renormalon singularities, and their implications for theoretical predictions, have been studied for some time now, and many of the ingredients of our discussion below can be found in refs.[1], and in the work of Mueller [2] in particular. Recently the effects of renormalons in predictions for the spectroscopy and decays of heavy quarks, obtained using the heavy quark expansion, have been studied [3]-[8]. We summarize in section 2 the picture which has emerged from these references for the appearance and cancellation of renormalon ambiguities, without reproducing the underlying arguments and derivations [3]. Our aim in this paper is to examine critically the procedure necessary to calculate the power corrections, and in particular to investigate whether higher-twist effects are sufficiently under control that their inclusion reduces the uncertainties in the theoretical predictions. Our conclusion is that to reach a precision of the order of the power corrections is likely to be a formidable task, and that the main limitation comes from the truncation of the perturbation series for the Wilson coefficients at low orders. The presentation below extends and clarifies that of our earlier paper [8].

We particularly wish to stress that, although some of the examples presented below are given using the lattice spacing as the cut-off, the problems discussed in this paper are completely general, and are not due to some peculiarity of the lattice regularization. An example, where the same problems are encountered as in the lattice theory, is given by the zero recoil inclusive sum rules [9, 10]. In this approach, in order to derive a bound on λ_1, an ultraviolet cut-off, Δ, is introduced on each side of the relation between the time-ordered product of two currents saturated with hadronic states and the corresponding quantity computed on quark states using the HQET. It is not surprising that perturbative corrections $\sim \alpha_s(\Delta)\,\Delta^2$ appear in the bound for λ_1, essentially eliminating the predictive power of the approach [10]. The appearance of such power divergences, and the consequent loss of precision when they are subtracted in a low order of perturbation theory, are general features in the evaluation of power corrections [8]. A more detailed discussion of this point will be given in subsec. 2.2.

The plan of the remainder of the paper is as follows. In the next section we briefly review the appearance of renormalon ambiguities in the Wilson coefficient functions of operator product expansions. The matrix elements of the higher twist or higher dimension operators have to be evaluated non-perturbatively. This can be done by comparing the theoretical expression for a physical quantity to its experimental value (where this is known) or by some non-perturbative method (such as lattice simulations). We show that in both cases the evaluation of power corrections requires the calculation of the perturbation series to

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[3] It should be stressed that although these arguments, based on the renormalization group, analyticity, and/or explicit calculations in the large N_f limit (N_f is the number of light quark flavours), are compelling, they nevertheless do not constitute a formal proof.

sufficiently high orders for the cancellation of renormalon ambiguities to be under control (sections 2.1 and 2.2 respectively). In sections 3-5 we study some simple examples, in order to investigate numerically the precision that might be reached in evaluating power corrections. We start by considering a toy model, which contains many of the general features concerning the appearance and cancellation of renormalon singularities in operator product expansions (section 3); we then proceed to the mass of a heavy quark, or equivalently the binding energy $\overline{\Lambda}$ (section 4), and to the determination of the gluon condensate and its use in phenomenological applications (section 5). Finally section 6 contains our conclusions.

2 Power Corrections

In this section the cancellation of renormalon ambiguities in the evaluation of hard scattering and decay processes is discussed [1].

Consider an operator $\hat{P}$ whose matrix element $\langle f|\hat{P}|i\rangle$ contains the non-perturbative effects for some physical process $\mathcal{P}$. In general $\hat{P}$ is non-local, for example it may be the T-product of two electromagnetic or weak currents at small separations (as in the e^+e^- annihilation cross-section or weak decays), or almost light-like separations (as in deep inelastic structure functions). In these cases $\hat{P}$ is expanded as a series of local operators, whose coefficients decrease as powers of the separation. In some important applications to heavy quark physics, QCD composite operators (represented by $\hat{P}$) containing the field of the heavy quark are expanded in terms of local operators of the HQET, with coefficients that decrease as inverse powers of the mass of the heavy quark.

The expansion of $\hat{P}$ (or the Fourier transform of $\hat{P}$) in terms of local operators $O_1(\mu)$, $O_2(\mu)$ etc., renormalized at a scale μ, takes the form:

$$\mathcal{P}_{fi}(Q^2) \equiv \langle\, f\,|\,\hat{P}(Q^2)\,|\,i\,\rangle = C_1(Q^2/\mu^2)\,\langle\, f\,|\,O_1(\mu)\,|\,i\,\rangle + \frac{C_2(Q^2/\mu^2)}{Q^n}\,\langle\, f\,|\,O_2(\mu)\,|\,i\,\rangle + O\left(\frac{1}{Q^{n+p}}\right)$$
(1)

where $n \geq 1$ and the coefficient functions C_i are independent of the states $|i\rangle$ and $|f\rangle$. Q is a large momentum scale; for example it may be the centre of mass energy in e^+e^- annihilation, the momentum transfer in deep inelastic scattering or the renormalization scale (of the order of the mass of the heavy quark) of the local operator $\hat{P}$ in heavy quark matrix elements. For clarity of notation, throughout this paper we suppress the dependence of the coefficient functions on the coupling constant, $\alpha_s(Q^2)$. We assume here that there is only one operator in each of the first two orders of the expansion. If this is not the case, then there is an additional mixing of operators, which requires only a minor modification of the discussion below. We will therefore not consider this possibility further. The final term of $O(1/Q^{n+p})$ in eq.(1) represents the contributions of operators of even higher dimension which will not be discussed here.

In some cases the operator O_2 is protected from mixing under renormalization with O_1, because of the presence of some symmetry. An important example of this in heavy quark physics is the chromomagnetic operator $\bar{h}\sigma_{\mu\nu}G^{\mu\nu}h$ (where h is the field of a static quark and $G^{\mu\nu}$ is the gluon field strength tensor), which cannot mix with the lower dimensional operator $\bar{h}h$ because it has a different spin structure. In such cases the corresponding problem of renormalon singularities does not arise. For the remainder of this paper we assume that O_2 and O_1 have the same quantum numbers, so that they can mix under renormalization. Another important exception to our general discussion is the difference of matrix elements of the kinetic energy operator taken between different hadronic states. In this case the higher dimensional operator, $\bar{h}\vec{D}^2h$, can mix with the lower dimensional one, $\bar{h}h$, but as the latter is a conserved current it has the same matrix element between all single hadron states. Thus the corresponding renormalon ambiguity cancels in the difference of matrix elements. In all of these exceptions the matrix elements of the higher dimensional operators (or linear combinations of matrix elements) are also the leading contribution to some physical quantity (such as the B^*-B mass splitting in the case of the chromomagnetic operator).

The usefulness of the operator product expansion in eq.(1) comes from the fact that the non-perturbative effects in the process $\mathcal{P}$ are contained in the matrix elements of the local operators O_i, whereas the coefficient functions C_i are calculable in perturbation theory. This fundamental property is threatened, however, by the presence of renormalon singularities in the Borel transform of the perturbation series of the coefficient functions, as we now explain.

We start by assuming that the operators on the right-hand side of eq.(1) are renormalized in some scheme based on the dimensional regularization of ultraviolet divergences, such as the $\overline{\text{MS}}$ scheme. Then the perturbation series for the coefficient function C_1 is divergent, and moreover is not Borel-summable, due to the presence of (infra-red) renormalon singularities in its Borel transform. Hence the "sum" of this series is not unique, the ambiguity being of $O(1/Q^n)$. This ambiguity is cancelled by that in the matrix element of the operator O_2, which arises as a result of an ultraviolet renormalon singularity in the Borel transform of the perturbation series of its matrix elements. This implies that renormalization schemes based on dimensional regularization do not define higher-twist or higher-dimensional operators, such as O_2, unambiguously, and alternative definitions have to be used. We now consider the two possible alternative approaches in turn. In the first of these, the problem of the ambiguity in the matrix elements of O_2 is eliminated by directly relating two, or more, physical quantities to which the matrix element of O_2 contributes (see section 2.1). In the second, the operator O_2 is defined using a hard ultraviolet cut-off and its matrix element is evaluated using some non-perturbative method, such as lattice simulations (see section 2.2).

2.1 Defining The Matrix Elements of O_2 Directly in Terms of Physical Quantities

In this subsection we show how to eliminate the renormalon ambiguity from the matrix elements of the higher-twist operator O_2 by sacrificing the possibility of making a theoretical prediction for one process, $\mathcal{P}_{fi}(Q^2)$ say. This is achieved by using its experimentally measured value, which we denote by $\mathcal{P}_{fi}^{exp}(Q^2)$. It is assumed here that the matrix element of O_1 is already known from some non-perturbative calculation, or from a second experimental measurement, or because, as happens in some important cases, it is a conserved operator or even the identity operator. We now show that such a procedure requires the control of the difference of two series which are not Borel-summable up to a precision of $O(1/Q^n)$. Imagine that we have computed the series for C_1 and C_2 in perturbation theory up to, and including, the term of $O(\alpha_s^k(Q))$. We take k such that the coefficients of the series for C_1 are not already diverging because of the renormalon singularity. Then we can define the matrix element of O_2 by the relation

$$\mathcal{P}_{fi}^{exp}(Q^2) = C_1^{(k)}(Q^2/\mu^2)\,\langle\, f\,|\,O_1(\mu)\,|\,i\,\rangle + \frac{C_2^{(k)}(Q^2/\mu^2)}{Q^n}\,\langle\, f\,|\,O_2^{(k)}(\mu)\,|\,i\,\rangle\,, \qquad (2)$$

where the superscript (k) on the coefficients denotes the fact that the perturbative series for C_1 and C_2 have been truncated at $O(\alpha_s^k(Q))$, and in the case of the matrix element of the higher-twist operator $\langle f|O_2^{(k)}(\mu)|i\rangle$ that its value, derived from a physical measurement combined with a perturbative calculation, depends on the order k. The definition of $\langle f|O_2^{(k)}(\mu)|i\rangle$, given in eq.(2) is, of course, totally unambiguous, although it does depend on Q and on the choice of process $\mathcal{P}_{fi}(Q^2)$, as well as on k. Different choices of these parameters can change the value of the matrix element of O_2 by terms of $O(\Lambda_{\mathrm{QCD}}^n)$ (or even larger terms if k is too small), i.e. by an amount which is of the same order as the expected value of the matrix element itself. We assume throughout this paper that the value of α_s is known to the required accuracy [4]. All the elements in eq.(2) are known except for the matrix element of O_2: $\mathcal{P}_{fi}^{exp}(Q^2)$ from experimental measurement, $\langle f|O_1(\mu)|i\rangle$ by symmetry, measurement or some non-perturbative method, and the remaining factors by perturbation theory up to $O(\alpha_s^k(Q^2))$. We now wish to use the value of $\langle f|O_2^{(k)}(\mu)|i\rangle$ defined in this way to make a prediction for another physical process, $\mathcal{R}_{fi}(Q^2)$ say [5], up to and including corrections of $O(1/Q^n)$:

$$\mathcal{R}_{fi}(Q^2) = D_1(Q^2/\mu^2)\,\langle\, f\,|\,O_1(\mu)\,|\,i\,\rangle + \frac{D_2(Q^2/\mu^2)}{Q^n}\,\langle\, f\,|\,O_2(\mu)\,|\,i\,\rangle + \cdots\,. \qquad (3)$$

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[4] Although this may present additional difficulties in practice.

[5] For simplicity we assume here that the scale Q^2 is the same in both processes; see below for the general case.

Returning to the example of $R_{e^+e^-}(Q^2)$ considered before, this corresponds to extracting the value of the gluon condensate from the cross-section at charmonium energies (the process $\mathcal{P}$), and using it together with parton model perturbative calculations to make predictions for other processes. If we know D_1 and D_2 at order k, then using eq.(2) as the definition for $\langle f|O_2^{(k)}(\mu)|i\rangle$ we may write

$$\mathcal{R}_{fi}(Q^2) = D_1^{(k)}\langle f|O_1|i\rangle + \frac{D_2^{(k)}}{Q^n}\langle f|O_2^{(k)}|i\rangle + \left[D_1^{(>k)} - C_1^{(>k)}\frac{D_2^{(k)}}{C_2^{(k)}}\right]\langle f|O_1|i\rangle + O\left(\frac{\alpha_s^{k+1}}{Q^n}\right), \quad (4)$$

where we have suppressed the renormalization scale (μ) and the arguments of the coefficient functions and of α_s. The superscript $(> k)$ denotes that the perturbation series for the coefficient function starts from the $O(\alpha_s^{k+1})$ term. Each of the terms on the r.h.s. of eq.(4) is free of renormalon ambiguities: the coefficients $D_1^{(k)}$ and $D_2^{(k)}$ because they correspond to series that have been truncated at a finite order in α_s; the matrix element $\langle f|O_2^{(k)}(\mu)|i\rangle$ because it has been defined directly from an experimental measurement using eq.(2); and the coefficient in the third term because the renormalon ambiguity cancels in the difference of the two series up to a precision of $O(1/Q^{n+p})$ (the perturbation series for the coefficient function D_1 also has a renormalon ambiguity of $O(1/Q^n)$, which is cancelled by that in the series for $C_1(D_2/C_2)$). The low order terms in these two series are in general very different, but the divergent behaviour at high orders is controlled by the same renormalon singularity and hence is the same.

The traditional procedure is to assume implicitly that the third term on the r.h.s. of eq.(4) is small, and to neglect it. For the small values of k for which perturbative results are generally available, there is in general no guarantee that the third term cannot give a contribution which is comparable to, or even larger than, that of the condensate itself. Moreover, the relative size of the two contributions will depend on the order k, the scale μ, and the process $\mathcal{P}_{fi}(Q^2)$ used to define the condensate, as well as on the process $\mathcal{R}_{fi}(Q^2)$ for which we want to make the prediction.

We have seen that in order for the prediction for $\mathcal{R}_{fi}(Q^2)$ to be accurate up to and including terms of $O(1/Q^n)$, the perturbative series for the coefficient functions must be known up to a sufficiently high order k, so as to make the third term on the r.h.s. of eq.(4) negligible. Formally, each term in these series is exponentially larger than the power corrections which are being calculated, and it may take a large number of terms before the required precision is achieved. In sections 3-5 we investigate the accuracy of the theoretical prediction as a function of the order in several simple cases.

In the above discussion we have assumed for simplicity that the two processes $\mathcal{P}_{fi}(Q^2)$ and $\mathcal{R}_{fi}(Q^2)$ occur at the same large momentum scale Q^2 [6]. This clearly is not necessary. For

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[6]We also assume that the coefficient functions for the two processes are known at the same order k.

example one might wish to predict the behaviour of the moments of deep inelastic structure functions with the photon's momentum Q^2, using an experimental measurement at a single value of Q^2 to determine the matrix element of O_2. Similar cancellations of renormalon ambiguities occur also in these cases, because the Q^2 dependence of the coefficient functions is given by perturbation theory.

2.2 Non-Perturbative Computation of the Matrix Elements of O_2

In this subsection, the procedure needed to compute the matrix elements of the higher-twist operator O_2 (non-perturbatively) is described. This requires the operators O_2 to be defined using a hard (dimensionful) ultraviolet cut-off Λ. For example, in lattice simulations it is natural to use bare operators, defined by the lattice action and with $\Lambda = a^{-1}$, where a is the lattice spacing. We therefore present the corresponding discussion in terms of bare operators, so that the renormalization scale μ is replaced by the cut-off Λ. We also have in mind that $\Lambda^2 \ll Q^2$, otherwise the use of the operator product expansion would not be necessary, one could just compute $\mathcal{P}_{fi}(Q^2)$ directly [7]. With a hard cut-off the operator O_2 mixes with O_1, with mixing coefficients which diverge as Λ^n. Thus, for the process $\mathcal{P}_{fi}(Q^2)$, we have

$$\mathcal{P}_{fi}(Q^2) = C_1^{(k)}(Q^2/\Lambda^2) \langle f \,|\, O_1(\Lambda) \,|\, i \rangle + \frac{C_2^{(k)}(Q^2/\Lambda^2)}{Q^n} \langle f \,|\, O_2(\Lambda) \,|\, i \rangle \,, \tag{5}$$

where the perturbation series, up to order k, for the coefficient function C_1 takes the form

$$C_1^{(k)}(Q^2/\Lambda^2) = c_1^{(k)}(Q^2/\Lambda^2) + \widetilde{c}_1^{(k)}(Q^2/\Lambda^2) \left(\frac{\Lambda}{Q} \right)^n \tag{6}$$

and the elements of the series c_1 and $\widetilde{c}_1$ diverge at most as powers of $\log(Q^2/\Lambda^2)$. The term proportional to $\widetilde{c}_1$ arises as a result of the mixing of O_2 with O_1 [8]. The matrix elements of O_2, computed with the hard cut-off, such as the lattice spacing, are well defined and unambiguous. Thus the same must be true for C_1, and the series in eq.(6) is indeed free of ambiguities of $O(1/Q^n)$. However this arises as a cancellation of the renormalon singularities in the two series $c_1^{(k)}$ and $\widetilde{c}_1^{(k)}$ as k becomes large. Each of these two series has a renormalon ambiguity of $O(1/Q^n)$. The cancellation of this ambiguity occurs between contributions which, in each order of perturbation theory, are of different order in $1/Q$. Again the low order terms in the two series are very different from each other, but the high order behaviour is governed by the same renormalon singularity and is the same. In order to predict $\mathcal{P}_{fi}(Q^2)$, we have to control the two series in eq.(6) up to a precision of better than $O(\Lambda_{\mathrm{QCD}}^n/Q^n)$.

A related practical problem is the cancellation of the power divergences of $O(\Lambda^n)$ in the coefficient function C_1 with those in the matrix element $\langle f \,|\, O_2(\Lambda) \,|\, i \rangle$. C_1 has to be computed

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[7]Although, in some cases this may not be possible in Euclidean space.

to sufficiently high order so that in eq.(5) this cancellation also occurs with a precision of better than $O(\Lambda_{\mathrm{QCD}}^n/Q^n)$.

In addition to the ultraviolet cut-off necessary to regularize the theory, it is useful in some processes to introduce a physical cut-off. This is the case, for example, in the zero-recoil sum rules [9, 10], where the cut-off, Δ, is defined to suppress the contribution of states with excitation energies greater than Δ [8] . There is a parallel between the above discussion with the hard cut-off Λ, introduced for the regularization, and that with the physical cut-off Δ. In the latter case, when, for example, the matrix element $\langle\, f\,|\, O_2(\Lambda)\,|\, i\,\rangle$ is the kinetic energy λ_1 (which is of $O(\Lambda_{\mathrm{QCD}}^2)$), the corresponding power correction to the coefficient function (i.e. the term proportional to $\widetilde{c}_1^{(k)}/Q^2$ in eq.(6)) contains perturbative corrections of $O(\alpha_s(\Delta)\,\Delta^2)$. The difficulty in achieving an accurate determination or bound with a physical cut-off follows the same discussion as for the hard ultraviolet cut-off. Only by arriving at an order such that $\alpha_s^n(\Delta) \leq O(\lambda_1/\Delta^2)$, can one obtain significant results.

2.3   Summary

Before concluding this section, we briefly summarize the main points of the above discussion. In order to evaluate the power corrections to hard scattering processes, it is necessary to determine the matrix elements of the higher-twist operators, such as O_2 in the above examples. This can be done by comparing a theoretical prediction which depends on a matrix element of O_2 to the experimenta data, or by computing the matrix element of O_2 using some non-perturbative method. We have argued that, in the predictions for physical quantities, renormalon ambiguities, which are of the same order as the power corrections being evaluated, cancel in the combinations of coefficient functions given in eqs.(4) and (6). Either of these procedures reduces the "intrinsic" error of the calculation from $O(1/Q^n)$ to $O(1/Q^{n+p})$, where by intrinsic we mean the minimum error achievable in principle. In order to reach the required precision, however, the series need to be evaluated to a sufficiently high order. An indication of whether the order is sufficiently high is given by the "common-sense" criterion that the last known term of the perturbative series of the leading coefficient functions should be significantly smaller than the power corrections. This point will be further illustrated in the next section, using a toy example, and in sections 4 and 5, where the presence and cancellation of renormalon ambiguities in quantities depending on the mass of the heavy quark and on the gluon condensate will be discussed.

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[8] Note that $\Delta \gg \Lambda_{\mathrm{QCD}}$ in order to be able to use perturbation theory in the parton sector; on the other hand we want Δ to be as small as possible in order to suppress the contribution of excited states and to reduce the uncertainty due to the perturbative contributions which are quadratic in Δ (see below).

3   A Toy Model

In this section we study a simple example, which contains many of the general features expected in operator product expansions, including the next-to-leading twist contributions, and the corresponding renormalon singularity. In this example, the "physical" quantity $\mathcal{P}$ is defined by

$$\mathcal{P}(x, \overline{\epsilon}) \equiv 10^3 \int_0^\infty du \, e^{-u/x} f(u) \, , \tag{7}$$

where

$$f(u) = \frac{1}{1 - 2u} - \frac{\epsilon^{(1-2u)}}{1 - 2u} \frac{1}{\Gamma(1 + 2u)} \, ; \tag{8}$$

$f(u)$ is the Borel transform of $\mathcal{P}(x, \overline{\epsilon})$, and $\overline{\epsilon}$ is a function of ϵ and x defined in eq.(10) below. The factor of 10^3 in eq.(7) is introduced for convenience. In eq.(8) the first term, expanded in powers of u and integrated over u as in eq.(7), generates the perturbation series of the leading coefficient function in x

$$\int_0^\infty du \, e^{-u/x} \frac{1}{1 - 2u} \;\; \rightarrow \;\; \int_0^\infty du \, e^{-u/x} \sum_{j=0}^\infty (2u)^j$$

$$= \;\; x + 2x^2 + \cdots + \frac{\Gamma(k)}{2} (2x)^k + \cdots . \tag{9}$$

The second term in eq.(8) corresponds to the matrix element of the higher-twist operator, and we shall refer to its contribution to $\mathcal{P}$ as the "condensate" contribution. Its strength is governed by the parameter $\overline{\epsilon}$, defined by

$$\epsilon = \overline{\epsilon} \, e^{-1/2x} \, , \tag{10}$$

which shows that ϵ is a term of order Λ_{QCD}/Q when the "coupling" $x \sim 1/\ln(Q^2/\Lambda_{\text{QCD}}^2)$ [9]. In physical cases, $\overline{\epsilon}$ is determined by the non-perturbative dynamics, so that $\mathcal{P}$ is only a function of x. In our toy model, however, we will treat $\overline{\epsilon}$ as a free parameter. Its natural value is of $O(1)$. The factor of $1/\Gamma(1 + 2u)$ has been introduced in order to make the integral in eq.(7) converge at large values of u (for small values of $\overline{\epsilon}$) or to improve the convergence. Other choices of such damping factors at large u would have been equally good for our purposes.

Both terms in eq.(8) exhibit a renormalon singularity at $u = 1/2$; this however cancels in $f(u)$, in a way which is analogous to the cancellation that is expected to occur in physical cases. Because of this singularity, the coefficients of the perturbation theory grow like a factorial. As a consequence, even for small values of x, the contribution of high orders

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[9]Similar models can be constructed to mimic the cases in which the power corrections are suppressed as $(\Lambda_{\text{QCD}}/Q)^n$, with $n > 1$.

		$\bar{\epsilon} = 100$	$\bar{\epsilon} = 10$	$\bar{\epsilon} = 1$	$\bar{\epsilon} = 0.1$
$\mathcal{P}(x = 0.07, \bar{\epsilon})$		72.23	82.33	84.92	199.57
k	t_k	s_k	"Value of Condensate" $- \mathcal{C}_k(x, \bar{\epsilon})$		
1	70	70	2.23	12.33	14.92	129.57
2	9.8	79.8	$-$ 7.57	2.53	5.12	119.77
3	2.74	82.54	$-$ 10.32	-0.21	2.38	117.02
4	1.15	83.70	$-$ 11.47	-1.36	1.23	115.87
5	0.65	84.34	$-$ 12.12	-2.01	0.58	115.23
6	0.45	84.79	$-$ 12.57	-2.46	0.13	114.77
7	0.38	85.17	$-$ 12.95	-2.84	-0.25	114.39
8	0.37	85.55	$-$ 13.32	-3.21	-0.62	114.02

Table 1: Perturbative and non-perturbative contributions to $\mathcal{P}$, for $x = 0.07$. The full result for $\mathcal{P}$ is given in the second row. t_k and s_k are the k-th term and the sum of the first k terms of the perturbation series (9). The condensate contributions $\mathcal{C}_k$ are defined in eq.(11).

diverges. We denote by k_{min} the order in x for which the magnitude of the k-th term of the series is the smallest one; k_{min} depends on x only.

In order to mimic the procedure one is forced to adopt in realistic cases, we proceed as follows:

i) For any choice of x and $\bar{\epsilon}$, the value of $\mathcal{P}(x, \bar{\epsilon})$ is obtained exactly from eq.(7) by numerical integration, and represents the "experimental" result (with no error).

ii) We assume that only k terms, with $k \leq k_{min}$, of the perturbation series (9) are known.

iii) We define the "condensate" $\mathcal{C}_k$ as

$$\mathcal{C}_k(x, \bar{\epsilon}) = \mathcal{P}(x, \bar{\epsilon}) - \frac{1}{2} \sum_{j=1}^{k} \Gamma(j) \, (2x)^j \, , \tag{11}$$

where by writing explicitly the arguments of $\mathcal{C}_k$ we recall that $\bar{\epsilon}$ is being treated as a free parameter. The subscript k is a reminder that the condensate was defined by subtracting k terms of the perturbation series.

iv) It is then envisaged that the values of the condensate obtained in this way are used to make predictions for another process, $\mathcal{R}$ say, as explained in subsection 2.1.

The stability of the value of $\mathcal{C}_k$ with k, and the comparison of its value to the k-th term of the perturbation series, monitors the precision that can be reached in predictions for physical quantities containing corrections of $O(\Lambda_{\mathrm{QCD}}/Q)$. As an illustration we present in table 1 the numerical results for $x = 0.07$ (for which $k_{min} = 8$) and for $\bar{\epsilon} = 100, 10, 1$ and 0.1. The

"physical" value of $\mathcal{P}(x = 0.07, \bar{\epsilon})$ for each of the values of $\bar{\epsilon}$ is given in the second row of the table. t_k is the k-th term of the perturbation series (9) and s_k is the sum of the first k terms, and they are tabulated up to $k = k_{min} = 8$. The remaining entries in the table are the values of the "condensates", $\mathcal{C}_k$, which one would deduce at each order of perturbation theory. We now comment on these, distinguishing between large, intermediate and small values of $\bar{\epsilon}$ in turn:

   i) <u>Large value of $\bar{\epsilon}$</u> ($\bar{\epsilon} = 100$):
      For $\bar{\epsilon} = 100$, the values of the condensate ($\mathcal{C}_k$) stabilize quickly, so that already for $k = 3$, $\mathcal{C}_3$ is significantly larger than the corresponding term in perturbation theory (t_3) and the sum of the higher order terms from t_4 to $t_{k_{min}}$. If this is the case for both $\mathcal{P}$ and $\mathcal{R}$ (in the notation of section 2.1) then the precision of the prediction for $\mathcal{R}$ is clearly improved by including the contribution from the condensate, provided that at least three terms in the perturbation series have been calculated.

   ii) <u>Intermediate value of $\bar{\epsilon}$</u> ($\bar{\epsilon} = 10$ and $\bar{\epsilon} = 1$):
      As $\bar{\epsilon}$ is decreased, one has to calculate more terms of the perturbation theory before the values of $\mathcal{C}_k$ stabilize. For $\bar{\epsilon} = 10$, the improvement in including the condensate contribution is, at best, marginal, even if five or six terms of the perturbation theory have been computed. For $\bar{\epsilon} = 1$, $\mathcal{P}$ is estimated accurately by perturbation theory, and the condensate contribution is too small to be determined. In many practical situations it may be sufficient to know that the condensate is smaller than some value. The uncertainty in the perturbation series is independent of the value of $\bar{\epsilon}$.

   iii) <u>Small value of $\bar{\epsilon}$</u> ($\bar{\epsilon} = 0.1$):
      For small values of $\bar{\epsilon}$ the situation becomes very unstable, and the contribution of perturbation theory, even if one includes all the terms up to $t_{k_{min}}$, is a poor approximation to $\mathcal{P}$. The reason can be understood by considering the series in x generated by each of the two terms in eq.(8). The coefficients of each of these two series grow like the factorial of k, but this factorial growth is cancelled in their sum. For very small values of ϵ, however, this cancellation will only begin to take effect at very high orders, in particular at values of k such that $k \gg k_{min}$, at which the perturbation series (9) is already diverging rapidly with k. It would be fascinating to find a realistic physical example corresponding to this case.

The conclusions which we draw from this simple example are as follows. Imagine that $\mathcal{P}(x, \bar{\epsilon})$ has been measured, and that k terms of the perturbation series have been calculated. A reasonable estimate of the perturbative contribution to $\mathcal{P}$ would then be $s_k \pm t_k$. If $\mathcal{C}_k \gg t_k$, then it makes sense to call $\mathcal{C}_k$ the condensate contribution, and to use it in predictions for other processes for which the perturbation series has similar properties and has been calculated to the same precision. The challenge in phenomenological applications is to demonstrate that this is the case. Otherwise the condensate is of the same order as the

uncertainty in the perturbation series (or smaller) and hence its value would depend on the process from which it is extracted. t_k serves as an estimate of the uncertainty in the value of the condensate.

There is one further important point which we wish to stress, i.e. the universality of the higher order corrections. Consider a set of processes $\mathcal{P}, \mathcal{R} \cdots$ for which the leading power correction is given by the same matrix element of a given higher-dimensional operator. The high order behaviour of all the leading coefficient functions is then dominated by the same infra-red renormalon, and is hence universal. This leads to the possibility that the error in the prediction for process $\mathcal{R}$, obtained by using the condensate $\mathcal{C}_k$ determined in process $\mathcal{P}$, may be smaller than the estimated uncertainty in the value of the condensate $\mathcal{C}_k$ itself. In order to see this, note that the definition of the condensate in eq.(11) implies that

$$t_{k+1}^{\mathcal{P}} + \mathcal{C}_{k+1} - \mathcal{C}_k = 0 \; , \tag{12}$$

where the superscript on $t_{k+1}^{\mathcal{P}}$ implies that this is the $(k+1)$-th term in the perturbation series for the process $\mathcal{P}$. The difference between the $(k + 1)$-th order and k-th order predictions for $\mathcal{R}$ is

$$t_{k+1}^{\mathcal{R}} + \mathcal{C}_{k+1} - \mathcal{C}_k + O\left(x\frac{\Lambda_{\mathrm{QCD}}}{Q}\right) \; , \tag{13}$$

where the last term represents perturbative, renormalon-free corrections to the power suppressed term. If the perturbation series for the two processes are both dominated by the same renormalon, and k is sufficiently large, then it may be that $t_{k+1}^{\mathcal{R}} \simeq t_{k+1}^{\mathcal{P}}$, and that the $(k + 1)$-th contribution to the prediction for $\mathcal{R}$ is smaller than the difference in the condensates, $\mathcal{C}_{k+1} - \mathcal{C}_k$. In practice, however, often only one or two terms of the perturbation series are known and it is unclear to what extent this property of universality will be useful in phenomenological applications.

4   The Pole Mass of a Heavy Quark

In this section the discussion of section 2 is applied to the computation of the heavy-quark mass in the HQET. Of course in practice we do not know the perturbation series for the coefficient functions to sufficiently high order to be able to study the numerical effects of the cancellation of renormalon ambiguities directly. For this reason we present a calculation performed in the limit of a large number of light quark flavours (N_f), or more precisely we perform the perturbative calculations keeping only the term with the highest power of β_0 at each order: $\beta_0 = 11 - 2/3N_f$ is the lowest order coefficient in the β-function.

It is possible to compute the renormalized mass of a heavy quark, defined at a large renormalization scale, $\mu \gg \Lambda_{\mathrm{QCD}}$, from the matrix elements of the HQET obtained with some

non-perturbative method, such as lattice simulations [11]. As an example we consider $\overline{m}$, the mass defined in the $\overline{\text{MS}}$ scheme at a renormalization scale $\mu = \overline{m}$:

$$\overline{m} = m^{\overline{\text{MS}}}(m^{\overline{\text{MS}}}) \ . \tag{14}$$

The computation requires the expansion of the propagator of the heavy quark in QCD (with the mass, wave-function renormalization and coupling constant defined in the $\overline{\text{MS}}$ scheme say), in terms of matrix elements of operators in the HQET. We will see that the coefficient function of the leading operator, C_1 of eq.(1), is indeed a difference of two series (as in eq.(6)), each of which has a renormalon ambiguity of $O(\Lambda_{\text{QCD}})$, the ambiguity cancelling in the difference. In the large N_f limit the terms of the series can be calculated to arbitrarily high orders, and the cancellation of the ambiguity then observed. This is done in subsection 4.2; in subsection 4.1 we start with a brief review of how one computes $\overline{m}$ using non-perturbative methods such as lattice simulations.

4.1   Evaluation of $\overline{m}$ in Lattice Simulations

In this subsection the procedure needed to evaluate $\overline{m}$ from simulations in the HQET up to, and including, terms of $O(\Lambda_{\text{QCD}})$, but neglecting terms of $O(\Lambda_{\text{QCD}}^2/\bar{m})$ is briefly reviewed [11] [10]. The non-perturbative quantity which is computed directly in lattice simulations, and which is required for the determination of $\overline{m}$, is the bare binding energy $\mathcal{E}_H$, where the label H denotes the hadron containing the heavy quark. $\mathcal{E}_H$ is obtained from the time dependence of the correlation function of two interpolating operators (J_H) for the hadron H:

$$\sum_{\vec{x}} \langle 0 \,|\, J_H(\vec{x}, t) \, J_H^\dagger(\vec{0}, 0) \,|\, 0 \rangle = Z \, e^{-\mathcal{E}_H t} \ , \tag{15}$$

where t is sufficiently large for the correlation function to be dominated by the lightest particle created by $J_H^\dagger$, which is assumed to be H.

The relation between $\mathcal{E}_H$ and $\overline{m}$ can be obtained by matching the heavy quark propagator in QCD with operator matrix elements evaluated in the HQET [11]. Using the $\overline{\text{MS}}$ renormalization scheme at a scale μ for the mass, wave function and coupling constant renormalization, the inverse propagator in QCD (S^{-1}) is of the form:

$$S_P^{-1}(v \cdot k) = m_Q - m(\mu) \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n c_n(m(\mu)/\mu)$$

$$+ v \cdot k \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n d_n(v \cdot k/\mu, m(\mu)/\mu) + O(\Lambda_{\text{QCD}}^2/\overline{m}) \tag{16}$$

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[10] In ref.[11] the generalization of this discussion to include terms of $O(\Lambda_{\text{QCD}}^2/\overline{m})$ is also presented.

where the momentum of the heavy quark is $m_Q v + k$ [11]; v is the four velocity of the heavy quark and S_P is defined by

$$\frac{1 + \not{v}}{2} S_P = \frac{1 + \not{v}}{2} S \frac{1 + \not{v}}{2} . \tag{17}$$

The first series on the right-hand side of eq.(16) is just the perturbative expansion of the pole mass of the heavy quark in terms of $m(\mu)$:

$$m_{\text{pole}} = m(\mu) \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n c_n(m(\mu)/\mu) . \tag{18}$$

The Borel transform of the series in eq.(18) has renormalon singularities, which is a manifestation of the fact that the pole mass is not a physical quantity [3, 4]. The ambiguity corresponding to the leading singularity is of $O(\Lambda_{\text{QCD}})$.

Now consider perturbation theory in the HQET (defined by the action $\bar{h}\, v \cdot D\, h$), using the lattice spacing as the ultraviolet cut-off. S_P^{-1} can be expressed in terms of S_{eff}^{-1}, the propagator in the HQET, which can be interpreted as the matrix element of the operator $O_2 = \bar{h}\, v \cdot D\, h$:

$$
\begin{aligned}
S_P^{-1}(v \cdot k) &= m_Q - (m_{\text{pole}} - \delta m) \\
&+ C_{v \cdot D}(m(\mu)a, m(\mu)/\mu) S_{\text{eff}}^{-1}(v \cdot k\, a) + O(\Lambda_{\text{QCD}}^2/m(\mu)) ,
\end{aligned} \tag{19}
$$

where the series

$$\delta m = \frac{4}{a} \frac{C_F}{\beta_0} \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} \frac{X_n}{a} \tag{20}$$

is just the perturbative expansion of the inverse of the quark propagator in the HQET at zero momentum. The leading renormalon singularity in m_{pole} is cancelled by the one in the series for δm, so that the combination $m_{\text{pole}} - \delta m$ has no renormalon ambiguity of $O(\Lambda_{\text{QCD}})$ [8]. The two series in eqs.(18) and (20) are an example of the series in eq.(6). Here $\Lambda = a^{-1}$, Q is the mass of the heavy quark and $n = 1$, i.e. we are calculating the $O(\Lambda_{\text{QCD}})$ correction to the mass of the heavy quark, which is one power of m smaller than the leading term. The operator O_1 is $\bar{h}h$, which is a conserved current in the HQET with matrix element equal to 1, and so it does not appear explicitly in eq.(19).

The mass $\overline{m}$ is obtained from the relation

$$m_{\text{pole}} - \delta m = M_H - \mathcal{E}_H , \tag{21}$$

where M_H is the physical mass of the hadron H, and by inverting the relation in eq.(18) between the pole and $\overline{\text{MS}}$ masses. In eq.(21), the linear divergence present in $\mathcal{E}_H$ is cancelled by that in the series δm, and the renormalon in m_{pole} is cancelled by that in δm, as explained above. In the following subsection we study the numerical cancellation of the renormalon ambiguity between m_{pole} and δm. Numerical results for $\overline{m}$ obtained in this way (but with the perturbative terms only computed to one-loop order) have been presented in [11, 12].

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[11]The precise definition of m_Q here can be conveniently chosen later.
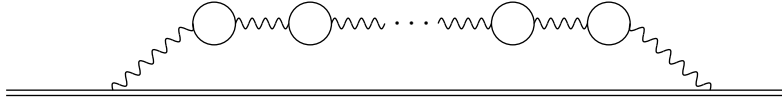
Figure 1: Diagrams which must be evaluated in order to study the heavy quark mass (and hence $\overline{\Lambda}$) in the large-β_0 limit. The double line represents the propagator of the heavy quark, and the bubbles represent light-quark loops.
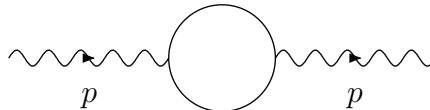

Figure 2: One-loop bubble graph contributing to the vacuum polarization of the gluon. This graph is the basic ingredient in the evaluation of the large-β_0 contribution to the quark mass (and other quantities). The solid lines represent light-quark propagators.

4.2 Cancellation of Renormalon Ambiguities in the Heavy-Quark Mass

In this subsection we explicitly trace how the cancellation of the renormalon ambiguities occurs in the combination $m_{\text{pole}} - \delta m$. The calculation is performed in the large-β_0 limit, in which only the terms containing the leading power of β_0 are kept in each order of perturbation theory. As a further simplification, we perform this calculation in the large-β_0 limit, with the Pauli-Villars (PV) cut-off Λ as the ultraviolet regulator in the effective theory (rather than the lattice spacing) [12]. In other words, in analogy with lattice field theory, we imagine that we have computed $\mathcal{E}_H$ non-perturbatively in the PV theory for some hadron H, and now perform the calculation of the matching term $m_{\text{pole}} - \delta m$ in perturbation theory.

In order to obtain the result in the lowest non-trivial order in the large-β_0 approximation, it is sufficient to evaluate the set of diagrams in fig. 1, summing the contributions from an arbitrary number of light-quark loops. Consider a single light-quark loop insertion as in fig. 2. We denote the expression from this diagram by $L(p, m_q^2)$, where m_q is the mass of the light quark. L is given in terms of a divergent integral, which we regulate by taking $L(p, 0) + L(p, 2\Lambda^2) - 2L(p, \Lambda^2)$ as the regulated expression for the diagram, as suggested in ref.[13]. Other choices are also possible. For the gluon propagator it is sufficient to take

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[12] The general features of the cancellation of the renormalon ambiguities are qualitatively the same with any hard cut-off. For the purposes of illustration the PV cut-off is very convenient.

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$-ig^{\mu\nu}(1/q^2 - 1/(q^2 - \Lambda^2))$ since any terms proportional to $q^\mu q^\nu$ in the gluon propagator do not contribute to δm. We then find

$$\delta m = 4\Lambda \frac{C_F}{\beta_0} \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\Lambda)\beta_0}{4\pi} \right)^{(n+1)} K_n \ , \tag{22}$$

where

$$K_n = \int_0^\infty \frac{dx}{(1+x^2)^{n+1}} \left[5/3 - \ln(2x^2) - F(x^2/2) + 2F(x^2) \right]^n \ , \tag{23}$$

$\alpha_s(\Lambda)$ is the bare coupling constant and the function

$$F(x^2) = 6 \int_0^1 dy \, y(1-y) \ln(1 + y(1-y)x^2) \tag{24}$$

can be readily evaluated. The position of the leading infra-red renormalon singularity is known, and by inverting the Borel transform using the saddle point method we find that the behaviour of K_n at large n is given by:

$$K_n \to \frac{e^{5/6}}{\sqrt{2}} 2^n n! \tag{25}$$

The numerical results obtained by using eq.(23) approximate their asymptotic values in (25) to better than 5% already for $n = 4$.

The relation between the pole mass and $\overline{m}$ in the large-β_0 limit is given by [14]:

$$m_{\text{pole}} = \overline{m} \left[1 + \frac{\alpha_s(\overline{m})C_F}{\pi} \left(1 + \sum_{n=1}^{\infty} d_n \left(\frac{\alpha_s(\overline{m})\beta_0}{4\pi} \right)^n \right) \right] \ , \tag{26}$$

where $\alpha_s(\overline{m})$ is the $\overline{\text{MS}}$ coupling constant at the renormalization scale $\overline{m}$. For large values of n, the coefficients d_n behave as

$$d_n \to e^{5/6} 2^n n! \tag{27}$$

and again this asymptotic relation is well satisfied by the numerical results for relatively small values of n.

To demonstrate the numerical cancellation of the leading renormalon singularity, we do the following. In the second column of table 2 we present the results for the pole mass obtained from eq.(26), with the assumption that $\overline{m} = 4.5$ GeV, and $\alpha_s(\overline{m}) = 0.2$. The results are presented in successive orders of perturbation theory in the $\overline{\text{MS}}$ coupling constant at $\mu = \overline{m}$. As expected, the value of m_{pole} increases rapidly at high orders due to the presence of the renormalon singularity. The standard approach when dealing with an asymptotic series is to consider only the first few terms to estimate the result. Since the smallest term in the series is of $O(100\,\text{MeV})$ (for $n = 3 - 5$), i.e. of $O(\Lambda_{\text{QCD}})$ as expected, this can be viewed

n	m_{pole} GeV	$m_{\text{pole}} - \delta m$ GeV
tree	4.5	4.5
0	4.88	4.62
1	5.12	4.72
2	5.24	4.74
3	5.34	4.76
4	5.44	4.77
5	5.58	4.77
6	5.80	4.78
7	6.21	4.78
8	7.08	4.79

Table 2: m_{pole} and $m_{\text{pole}} - \delta m$ calculated up to $O(\alpha_s^{n+1}(\overline{m}))$ in perturbation theory, in the large-β_0 limit, using $\overline{m} = 4.5$ GeV and $\alpha_s(\overline{m}) = 0.2$.

as the intrinsic uncertainty in the unphysical quantity m_{pole}. In the third column of table 2 we present the analogous results for $m_{\text{pole}} - \delta m$, also expanded in terms of the $\overline{\text{MS}}$ coupling constant at $\mu = \overline{m}$. In eq.(22) we have taken $\Lambda = 2$ GeV, and have expanded the Pauli-Villars coupling $\alpha_s(\Lambda)$ in terms of the $\overline{\text{MS}}$ coupling $\alpha_s(\overline{m})$. The series of the difference $m_{\text{pole}} - \delta m$ stabilizes at lower orders because of the cancellation of the leading renormalon singularity. The remaining uncertainty is now much smaller (in general it would be of $O(\Lambda_{\text{QCD}}^2/\overline{m})$, but in the large-$\beta_0$ limit the corresponding renormalon is absent [4, 15], so that the ambiguity is of $O(\Lambda_{\text{QCD}}^3/\overline{m}^2)$). As a consequence the smallest term in the series, which sets the scale of the intrinsic uncertainty, is now less than 10 MeV.

The unphysical parameter $\overline{\Lambda} = M_H - m_{\text{pole}}$ is frequently used in phenomenological studies of B-physics. Results and bounds for $\overline{\Lambda}$ are presented (see for example [16] and the reviews [17, 18], and references therein). In order for this to make any sense, $\overline{\Lambda}$ must be defined precisely in terms of some physical quantity, and its value will depend on this physical quantity and on the order of perturbation theory used to extract $\overline{\Lambda}$. As can be seen in table 2, the values for $\overline{\Lambda}$ will change by several hundred MeV as the order of perturbation theory is increased. This also implies that when using the value of $\overline{\Lambda}$ measured in one process to make predictions for a second one, we should use the same order of perturbation theory in both processes and hope that the universality discussed at the end of section 3 holds to a good approximation.

All calculations of power corrections to hard scattering and decay processes will involve a cancellation of renormalon ambiguities similar to the one discussed in this section [13]. Of course, one should remember that the results in table 2 were obtained using an approximation and can only be taken as being indicative. Nevertheless they highlight the difficulty of

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[13] Unless there is a symmetry which prevents the mixing of the operators with different dimensions.

evaluating power corrections. If one's aim is to try to evaluate $\overline{m}$ up to uncertainties of $O(\Lambda_{\mathrm{QCD}}^2/\overline{m}) \simeq 25$ MeV or so, many orders of perturbation theory would be required because the series for $m_{\mathrm{pole}} - \delta m$ converges very slowly. In the lattice theory only the one-loop term is known (corresponding to $n = 0$ in table 2), and the continuum relation (18) is known up to two-loop order. This would suggest that values for the binding energy $\overline{\Lambda}$ which are used in phenomenological studies in heavy-quark physics are uncertain by an amount of order 100 MeV, due to our ignorance of the higher order perturbative terms. A similar comment applies to our computation of $\overline{m}$ in ref.[11] where this uncertainty was probably underestimated, and to the result in [12].

It is likely that the case of $\overline{\Lambda}$ is a relatively good one, since the corrections are suppressed by just one power of the mass of the heavy quark, and, at least in the large-β_0 approximation, the asymptotic behaviour (25) and (27) seems to set in at low orders of perturbation theory. In the next section we consider an important example where this is not the case, that of the evaluation of the gluon condensate.

5 The Gluon Condensate

In this section we study another important example, that of the contribution of the gluon condensate to physical quantities in general, and to the D-function in e^+e^- annihilation in particular. The D-function is defined by $D(Q^2) = -1/4Q^2 d\Pi(Q^2)/dQ^2$, where Π is obtained from the correlation function of two electromagnetic currents:

$$i \int d^4x \, e^{iq \cdot x} \langle 0| \, T\{J_\mu(x) J_\nu(0)\} \, |0\,\rangle = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(-q^2) \tag{28}$$

and $Q^2 = -q^2$. The gluon condensate, $\langle \alpha_s G^2/\pi \rangle$, is the vacuum expectation value of an operator of dimension 4, and hence its contribution is of $O(\Lambda_{\mathrm{QCD}}^4/Q^4)$ relative to the perturbative terms. As we have tried to stress throughout this paper, the gluon condensate itself is not a physical quantity, as it contains a renormalon ambiguity of $O(\Lambda_{\mathrm{QCD}}^4)$. This ambiguity is cancelled by that in the perturbation series for the D-function (or, in general, by that in the leading coefficient function for the process being studied). The suppression by four powers of Q implies that the cancellations are very large and leads to enormous difficulties in the quantitative evaluation of the power corrections.

In subsection 5.2 below we study the high order behaviour of the perturbative series for the D-function in the large β_0 limit. We argue that even at low values of Q^2, where the relative contribution of the power corrections is significant, the uncertainty and ambiguity in the perturbation series can be comparable to the contribution normally ascribed to the condensate. We start, however, by a discussion of the computation of the gluon condensate in lattice simulations. We demonstrate that the extremely large numerical cancellations which

arise in the subtraction of the quartic power divergence (i.e. of the terms which diverge as a^{-4}), and the presence of the corresponding renormalon singularity, make the quantitative evaluation of the leading power corrections (i.e. the $O(1/Q^4)$ corrections) prohibitatively difficult.

5.1 Evaluation of the Gluon Condensate in Lattice Simulations

A natural definition of the condensate in lattice QCD is given in terms of the expectation value of the plaquette variable $P_{\mu\nu}$ (μ and ν define the plane containing the plaquette, but the expectation value is, of course, independent of the choice of plane):

$$P \equiv \langle\, 1 - \frac{1}{3}\mathrm{Tr}P_{\mu\nu}\,\rangle \equiv \frac{\pi^2}{36}a^4\,\langle\,\frac{\alpha_s}{\pi}G^2\,\rangle_{\mathrm{latt}}\;,\tag{29}$$

where a is the lattice spacing and the subscript stands for "lattice". The variable P is measured very precisely in lattice simulations for the Wilson action, at all standard values of the lattice spacing (as well as for some other lattice discretizations of QCD). In lattice QCD, as with any regularization using a hard cut-off, P is not zero in perturbation theory, but is given by an expansion of the form

$$P = \sum_{n=1} \frac{c_n}{\beta^n}\;,\tag{30}$$

where $\beta = 6/g_0^2(a)$ and $g_0(a)$ is the bare lattice coupling constant. The series in eq.(30) arises as a result of the mixing of the G^2 with the identity operator. The first 8 (!) coefficients c_i have been obtained numerically using Langevin techniques [19]. We would like to use the computed value of P and perturbative matching to calculate the $O(\Lambda_{\mathrm{QCD}}^4/Q^4)$ corrections to some physical process. For example for the D-function the relation is:

$$D(Q^2) = D_{\mathrm{pert}}^{\mathrm{cont}} - D_{\mathrm{pert}}^{\mathrm{latt}} + \frac{24}{a^4Q^4}\left(1 + \frac{7}{6}\frac{\alpha_s^{\overline{\mathrm{MS}}}(Q)}{\pi} + \cdots\right)P\;,\tag{31}$$

where

$$D_{\mathrm{pert}}^{\mathrm{cont}} = 1 + \frac{\alpha_s^{\overline{\mathrm{MS}}}(Q)}{\pi} + 1.640\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}(Q)}{\pi}\right)^2 + \cdots\tag{32}$$

and

$$D_{\mathrm{pert}}^{\mathrm{latt}} = \frac{24}{a^4Q^4}\left(1 + \frac{7}{6}\frac{\alpha_s^{\overline{\mathrm{MS}}}(Q)}{\pi} + \cdots\right)\sum_{n=1}\frac{c_n}{\beta^n}\;.\tag{33}$$

$D_{\mathrm{pert}}^{\mathrm{cont}}$ is the perturbative series for the D-function and the superscript stands for "continuum". The series $D_{\mathrm{pert}}^{\mathrm{latt}}$ arises from the matching of the D-function with the lattice operator

in eq.(29), and the superscript stands for "lattice". In the notation of subsection 2.2, and eqs. (5) and (6) in particular,

$$D_{\text{pert}}^{\text{cont}} = c_1(\Lambda^2/Q^2)\,, \tag{34}$$

and

$$D_{\text{pert}}^{\text{latt}} = -\tilde{c}_1(Q^2/\Lambda^2)\left(\frac{\Lambda}{Q}\right)^n\,, \tag{35}$$

with $\Lambda = a^{-1}$ and $n = 4$. In this case the operator O_1 is the identity operator. The leading infra-red renormalon singularity in the series $D_{\text{pert}}^{\text{cont}}$ is cancelled by the renormalon in $D_{\text{pert}}^{\text{latt}}$, as explained in subsection 2.2.

As an example of how serious the cancellations are, and how very difficult it is to obtain results with the required precision we take $Q = m_\tau$ (m_τ is the mass of the τ-lepton), $\beta = 5.7$, which corresponds to an inverse lattice spacing of about 1.15 GeV, and $\alpha_s^{\overline{\text{MS}}}(m_\tau)$ $= 0.32$. We are forced to choose fairly large values of Q^2, for which the contribution of the condensate is expected to be small, since in order to make use of the lattice results we require $\Lambda_{\text{QCD}} < a^{-1} < Q$. We consider $\beta = 5.7$ ($a^{-1} \simeq 1.15\,\text{GeV}$) to be about the smallest value of β (and inverse lattice spacing) for which one may reasonably expect that lattice artefacts will not invalidate the interpretation of the results. For these values of the parameters, the three components in eq.(31) begin like:

$$D_{\text{pert}}^{\text{cont}} = 1 + 0.102 + 0.017 + \cdots \tag{36}$$

$$D_{\text{pert}}^{\text{latt}} = 5.630 - 6.259 + \cdots \tag{37}$$

$$\frac{24}{a^4 Q^4}\left(1 + \frac{7}{6}\frac{\alpha_s^{\overline{\text{MS}}}(Q)}{\pi} + \cdots\right)P = (1 + 0.119 + \cdots)\,1.894\,. \tag{38}$$

The numbers in eq. (37) have been obtained after rewriting the series in terms of $\alpha_s^{\overline{\text{MS}}}(Q)$. We see that the terms in eqs.(37) and (38) are huge compared to the contribution one would normally ascribe to the gluon condensate of about 1% or so (taking $\langle\,\alpha_s G^2/\pi\,\rangle \simeq 0.018\,\text{GeV}^4$). It would clearly be enormously difficult to quantify these power corrections accurately. This would require the perturbation series $D_{\text{pert}}^{\text{cont}} - D_{\text{pert}}^{\text{latt}}$, and also the relation between the lattice and $\overline{\text{MS}}$ coupling constants, to be known to extremely high orders.

Although the numerical results presented above were obtained using the plaquette variable to define the gluon condensate on the lattice as in eq.(29), the general discussion applies to any choice of operator. The problems arise from the presence of quartic power divergences and renormalon ambiguities, which are general features of lattice attempts to evaluate the power corrections associated with the gluon condensate.

In ref.[20] the author has attempted to define the gluon condensate from $P - \sum c_n/\beta^n$, where P has been measured numerically, and the first 8 coefficients c_i are known [19]. He uses different resummation techniques for the perturbative terms and finds results for the

condensate which depend significantly on the method of summation, and are always at least five times larger than those used in phenomenological applications. The point that we are trying to stress in this paper is that such an analysis is theoretically inconsistent. The series $\sum c_n/\beta^n$ has a renormalon ambiguity of $O(\Lambda_{\text{QCD}}^4)$ (which, as always, is of the same order as the effect one is trying to evaluate); thus subtracting the perturbative series $\sum c_n/\beta^n$ from the computed values of P leaves an intrinsic arbitrariness of this order. In order to eliminate this arbitrariness we have to include the measured value of P in the prediction for a physical quantity using the matching procedure described above. In this way, for the D-function (which is a typical example) the renormalon ambiguity cancels between the first two terms on the right-hand side of eq. (31).

5.2   The Uncertainty in the Perturbation Series for the D-function

Although the discussion in the previous subsection was concerned specifically with the evaluation of the power corrections to the D-function using lattice simulations, we believe that it is also very difficult to control the corresponding calculations in phenomenological studies using continuum regularizations. Consider the perturbation series for the D-function using the $\overline{\text{MS}}$ coupling constant:

$$D_{\text{pert}}^{\text{cont}} = 1 + \frac{\alpha_s(Q^2)}{\pi} + 1.6398 \left(\frac{\alpha_s(Q^2)}{\pi} \right)^2 + 6.37101 \left(\frac{\alpha_s(Q^2)}{\pi} \right)^3 + \cdots . \tag{39}$$

The coefficient of $(\alpha_s(Q^2)/\pi)^4$ has been estimated to be about 27.5 [21], based on calculations using the principle of minimal sensitivity [22] and the effective charge approach [23]. For values of Q^2 such that $\alpha_s(Q^2) < 1/2$ say, the series appears to be reasonably well behaved. However, in order to gain some insight into the effects of the infra-red renormalon in the series $D_{\text{pert}}^{\text{cont}}$ we have to go beyond the order for which the coefficients are known. For this reason we study this series in the large-β_0 limit, for which the coefficients have been determined in refs. [24, 25]. Writing

$$D_{\text{pert}}^{\text{cont}} = 1 + \frac{1}{\beta_0} \sum_{n=1}^{\infty} \kappa_n \left(\frac{\beta_0 \alpha_s(Q^2)}{\pi} \right)^n , \tag{40}$$

the coefficients κ_n can readily be obtained from the Borel transform of the series,

$$\sum_{n=1}^{\infty} \frac{(4u)^{n-1}}{\Gamma(n)} \kappa_n = \frac{32 \, e^{-Cu}}{3(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k \, k}{[k^2 - (1-u)^2]^2} , \tag{41}$$

where in the $\overline{\text{MS}}$ scheme $C = -5/3$. The large order behaviour of the series $D_{\text{pert}}^{\text{cont}}$ is dominated by the singularity closest to the origin, which in this case is a double (ultraviolet) renormalon pole at $u = -1$. The high order terms generated by this pole diverge like a factorial of the order, but with alternating signs, corresponding to a behaviour which is Borel-summable.

n	s_n	t_n	$t_{n,\text{asymp}}$
0	1	1	1
1	1.049	0.049	4.461
2	1.089	0.040	0.799
3	1.125	0.036	0.286
4	1.161	0.036	0.154
5	1.201	0.039	0.110
6	1.248	0.047	0.099
7	1.310	0.062	0.106
8	1.399	0.090	0.133
15	23.511	13.016	13.536

Table 3: Values of s_n, t_n and $t_{n,\text{asymp}}$ for a value of Q^2 such that $\alpha_s(Q^2) = 1/2$ ($Q \simeq 0.75$ GeV).

Thus although the presence of ultraviolet renormalons may add further practical difficulties to the evaluation of the power corrections of $O((\Lambda_{\text{QCD}}/Q)^4)$, we will not consider them further here. Specifically, we subtract the contributions of the ultraviolet poles at $u = -1$ and $u = -2$, which appear in the terms with $k = 2$ and 3 respectively in eq. (41), by considering the behaviour of the coefficients κ'_n obtained from

$$\sum_{n=1}^{\infty} \frac{(4u)^{n-1}}{\Gamma(n)} \kappa'_n = \frac{32 e^{-Cu}}{3(2-u)} \left\{ \frac{90 - 39u + 5u^2}{144(3-u)^2} - \frac{224 - 72u + 7u^2}{576(4-u)^2} + \sum_{k=4}^{\infty} \frac{(-1)^k \, k}{[k^2 - (1-u)^2]^2} \right\} . \tag{42}$$

The residues of the infra-red renormalons (at $u = 2$ and above) are the same in eqs.(41) and (42). The large order behaviour of the coefficients κ'_n is given by

$$\kappa'_n \to \kappa'_{n,\text{asymp}} = \frac{e^{10/3}}{8^{n-1}} (n-1)! , \tag{43}$$

as $n \to \infty$.

Consider the perturbation series generated by the coefficients κ'_n, $D_{\text{pert}}^{\text{cont}} = \sum_{n=0} t_n$, where $t_0 = 1$ and

$$t_n = \frac{\kappa'_n}{\beta_0} \left(\frac{\beta_0 \alpha_s(Q^2)}{\pi} \right)^n \tag{44}$$

for $n \geq 1$. We denote by s_n the sum of the series up to n-th order, $s_n = \sum_{k=0}^{n} t_k$. It is also convenient to define $t_{n,\text{asymp}}$ as in eq.(44), but with κ'_n replaced by the asymptotic form $\kappa'_{n,,\text{asymp}}$.

In order to illustrate the difficulties of evaluating the perturbation series with sufficient precision to make the inclusion of the corrections of order $(\Lambda_{\text{QCD}}/Q)^4$ meaningful, we present

an example. In table 5.2 we give the values of s_n, t_n and $t_{n,\text{asymp}}$ obtained for a value of Q^2 such that the $\overline{\text{MS}}$ coupling constant $\alpha_s(Q^2) = 1/2$ ($Q \simeq 0.75$ GeV). We choose a small value of Q^2 so as to enhance the contribution of the power corrections (it would be even more difficult to quantify the power corrections at higher values of Q^2). We now make some comments on these results:

i) The smallest term in the series $\{t_n\}$ occurs for $n = 4$, $t_4 = 0.036$. One might therefore be tempted to take the four-loop result, $s_4 = 1.161$, as the best estimate for the sum of the perturbation theory, and $t_4 \sim 4\%$ as the estimate of the renormalon ambiguity. One might also expect that the inclusion of the gluon condensate will eliminate this uncertainty (up to a precision of order $(\Lambda_{\text{QCD}}/Q)^6$). Moreover since with standard phenomenological values of the condensate ($\langle \alpha_s G^2/\pi \rangle \simeq 0.018 \, \text{GeV}^4$) it is expected that its contribution to the D-function at such low values of Q^2 is about 40%, the ambiguity of about 4% can be considered negligible.

Such an interpretation is wrong, however. The perturbation theory for $n \simeq 4$ is not yet dominated by the leading infra-red renormalon (as can be seen, for example, from the fact that t_4 is very different from $t_{4,\text{asymp}}$, or by evaluating the contribution from the next-to leading renormalon at $u = 3$). The smallest term in the series $\{t_{n,\text{asymp}}\}$ occurs at $n = 6$ and is about 10%. Even for $n \simeq 6$, however, the perturbation series $\{t_n\}$ is not well approximated by $\{t_{n,\text{asymp}}\}$. Thus it is not easy to estimate the uncertainty in the evaluation of the perturbation series, other than to say that it is certainly greater than 10%.

ii) Since the perturbation series approaches its asymptotic value very slowly, there is no reason why the low order terms should be approximately universal. For example, the large order contribution to κ'_n from the infra-red renormalon at $u = 3$ is $-4/27 \, e^5 \, n!/12^{n-1}$ and for $n \leq 4$ is greater than or comparable to that of the leading infra-red renormalon at $u = 2$ (which is $e^{10/3} \, (n-1)!/8^{n-1}$, see eq.(43)). The renormalon at $u = 3$ corresponds to operators of dimension 6, whose contribution relative to the gluon condensate depends on the process, and hence the low order terms of the perturbation series are not universal [14].

iii) The discussion in i) and ii) was based on the assumption that it is possible to calculate many orders of perturbation theory and to study the extent to which the asymptotic behaviour has been reached. Of course, in practice, usually only one or two terms of the perturbation theory are known, which adds substantially to the uncertainty. For example the values of the contribution from the "condensate", which one would obtain by subtracting either the one-loop result (s_1) or the six-loop one (s_6) from the

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[14] A related question is whether at such low scales, the non-perturbative contibutions of the operators corresponding to $u = 3$ and above do not become as large as that of the gluon condensate.

23

measured value of $D(Q^2)$, would differ by about 0.2, i.e. by 20% of the D-function itself.

iv) It may be the case that for some processes, and at small values of Q^2 in particular, the "condensate" contribution is much larger than all the combined uncertainties. In the language of subsection 2.1 this would be necessary both for the physical quantity $\mathcal{P}$ being used to determine the condensate (e.g. some correlation function used in the study of the spectrum of charmonium) and for the quantity $\mathcal{R}$ for which the prediction is being made (e.g. decay constants or semileptonic form factors of heavy mesons). This example demonstrates, however, that to be confident that the uncertainties are indeed sufficiently small will require considerable effort.

In this simple example the uncertainty and ambiguity in the perturbation series for $D(Q^2)$ is at least a significant fraction (50–100% ?) of the expected size of the leading power correction. This is in spite of the fact that Q^2 was chosen to be small in an attempt to minimize the relative size of the ambiguity. Although one can change the details of the discussion by using different values of Q^2, different renormalized coupling constants as the expansion parameters, or different physical processes, it is our contention that the difficulties discussed above are general and cannot be easily overcome.

6 Conclusions

In this paper we have studied several examples in order to understand whether it is possible to compute power corrections to hard scattering and decay processes to a sufficient level of precision. By this we mean that the theoretical uncertainties must be smaller than the power corrections themselves. In all of the examples considered, we have found that this is not possible unless we are able to control perturbation theory at higher orders than those available at present. Since the arguments discussed in this paper are general, and not specific to the examples used, we believe that even the leading power corrections are currently not well determined, and there is little hope to compute higher order power corrections [15]. We hope that these disappointing conclusions and provocative comments will help to stimulate further debate and a systematic investigation of this central question of particle physics phenomenology.

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[15]For exceptions to the general discussion see section 2.

Acknowledgements

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